Introduction to Mathematical Quantum Theory

Text of the Exercises

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Exercise 1

Let ψ be a unit vector in $L^2(\mathbb{R})$ such that $x\psi,\ x^2\psi \in L^2(\mathbb{R})$. Prove that

$$\langle X^2\rangle_\psi \geqslant \left(\langle X\rangle_\psi\right)^2, \tag{1}$$

where as we defined in class, X is the operator given by the multliplication by x and

$$\langle A\rangle_\psi := \langle \psi, A\psi\rangle. \tag{2}$$

Hint: Use Jensen inequality.

Exercise 2

Let $\alpha := \{\alpha_n\}_{n\in\mathbb{Z}}$ be a sequence of complex numbers. Consider the Hilbert space of the square integrable functions $\mathfrak{h} := l^2(\mathbb{Z})$. Consider the operator that to the sequence $x := \{x_n\}_{n\in\mathbb{Z}}$ associate the sequence $M_\alpha x = \{\alpha_n x_n\}_{n\in\mathbb{Z}}$.

Suppose that $\|\alpha\|_\infty := \sup_{n\in\mathbb{Z}} |\alpha_n| < +\infty$. Prove that M_a is a well defined linear bounded operator from $\mathfrak{h}$ to itself and prove that $\|M_\alpha\| = \|\alpha\|_\infty$.

Exercise 3

Consider the Hilbert space $\mathfrak{h} := L^2(\mathbb{R})$. And the operator H define

$$\mathcal{D}(H) := H^2(\mathbb{R}) = \left\{\psi \in L^2(\mathbb{R}) \mid k^2\widehat{\psi} \in L^2(\mathbb{R})\right\}$$
$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(X),$$

where the operator $(V(X)\psi)(x) = V(x)\psi(x)$, with

$$V(x) := \begin{cases} -C & \text{if } |x| \leqslant A, \\ 0 & \text{if } |x| > A, \end{cases} \tag{3}$$

and with A and C positive constants. Consider $E \in (-\infty, -C]$ and prove that there is no nonzero $\psi_E \in \mathcal{D}(H)$ such that

$$H\psi_E = E\psi_E. \tag{4}$$

Exercise 4

Let $\mathfrak{h}$, H and $\mathcal{D}(H)$ as in Exercise 3. In class we saw that for any $E \in (-C, 0)$ there is always at least one nonzero even solution ψ_E to the problem $H\psi_E = E\psi_E$.

Prove that if $A\sqrt{2mC}\hbar \leqslant \frac{\pi}{2}$ there are no nonzero odd solutions, and for larger values of C there is always at least one.